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**Abstract**
We present a general procedure for determining possible (nonuniform) magnetic fields such that the Pauli equation becomes quasi-exactly solvable (QES) with an underlying $sl(2)$ symmetry. This procedure makes full use of the close connection between QES systems and supersymmetry. Of the ten classes of $sl(2)$-based one-dimensional QES systems, we have found that nine classes allow such construction.

PACS numbers: 03.65.−w, 03.65.Fd, 02.30.Zz, 02.70.Hm

## 1. Introduction

The Pauli equation describes the motion of a charged particle in an external magnetic field. It is given by the Hamiltonian ($\hbar = e = 2m_e = c = 1$)

$$H = (p_x + A_x)^2 + (p_y + A_y)^2 + \frac{g}{2}(\nabla \times \mathbf{A})_z \sigma_z \tag{1}$$

where $p_x$, $p_y$ are the momentum operators, $g = 2$ is the gyromagnetic ratio, $\mathbf{A}$ is the vector potential of the electromagnetic field and $\sigma_z$ is the Pauli matrix. For uniform magnetic field $B_x = B_y = 0$, $B_z = B$ the system is exactly solvable, giving the Landau levels. On the other hand, it was proved in [1] that for any general magnetic field $B_z = B(x, y)$ perpendicular to the $xy$ plane, the ground state is exactly calculable, owing to the existence of supersymmetry (SUSY) in equation (1) [2]. The general result of [1] can be viewed as a very special case of the newly discovered quasi-exactly solvable (QES) systems, which are systems that allow parts of their spectrum to be solved algebraically [4–13].

The Landau problem and the result of [1] represent the two extremes of the spectral problem of equation (1). It is thus of interest to determine if other possibilities exist. Based on SUSY of the Pauli equation and the idea of shape invariance, it was shown that there exist three other forms of (nonuniform) magnetic field which make the Pauli equation exactly solvable [2] (see also [3]). It seems difficult, if not impossible, to find other forms of the magnetic field such that the Pauli equation could be exactly solved. A more modest aim is to determine magnetic fields such that parts of the spectrum of the Pauli equation can be algebraically obtained. In other words, one looks for those magnetic fields under which the Pauli equation becomes

quasi-exactly solvable (we note that for certain nonuniform magnetic fields, the Pauli equation admits two solutions [14] and thus can be considered as a QES system). But even with this modest aim, the possibility seems enormous, since there are many QES systems based on different Lie algebras. In this paper we would like to make an attempt in this direction based on the simplest Lie algebra, namely, the $sl(2)$ algebra. QES systems based on $sl(2)$ algebra have been completely classified by Turbiner [5], and the necessary and sufficient conditions for the normalizability of the wavefunctions in such systems were completely determined in [6]. It turns out that general forms of the magnetic field can be found so that the Pauli equation can be fitted into nine of the ten classes in [5]. The magnetic fields giving rise to these nine classes of QES potentials are divided into two groups: six in asymmetric gauge and the other three in symmetric gauge. We shall describe these cases separately in sections 2 and 3. We would like to mention here that it is not necessary to consider magnetic fields that give rise to QES Pauli Hamiltonians with periodic potentials, since it has been proved in [6] that the wavefunctions in such systems are not normalizable.

## 2. Magnetic field in asymmetric gauge

Consider the magnetic field in the asymmetric gauge given by the vector potential

$$A_x(x, y) = 0 \qquad A_y(x, y) = -\bar{W}(x) \tag{2}$$

where $\bar{W}(x)$ is an arbitrary function of $x$. The magnetic field **B** has components $B_x = B_y = 0$ and $B_z = B(x) = -\bar{W}'(x)$. The Pauli Hamiltonian is then given by

$$H = p_x^2 + (p_y - \bar{W}(x))^2 - \bar{W}'(x)\sigma_z \tag{3}$$

where $\bar{W}'(x) = \mathrm{d}\bar{W}(x)/\mathrm{d}x$. The eigenfunction $\tilde{\psi}$ can be factorized as

$$\tilde{\psi}(x, y) = \mathrm{e}^{-iky}\bar{\psi}(x). \tag{4}$$

Here $k$ ($-\infty < k < \infty$) are the eigenvalues of $p_y$, and $\bar{\psi}(x)$ is a two-component function of $x$. The upper and lower components of $\bar{\psi}$ are then governed by the Hamiltonians $H_-$ and $H_+$ respectively, where

$$H_{\mp} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (\bar{W}(x) + k)^2 \mp \bar{W}'(x). \tag{5}$$

In this form the SUSY structure of the Pauli equation is clearly exhibited, with $W(x) = \bar{W}(x) + k$ playing the role of the superpotential. Once the upper component $\psi$ of $\bar{\psi}$ is solved for a nonzero energy, the lower component can be obtained by applying an appropriate supercharge on the upper component [2] and vice versa. Hence, it is sufficient to consider the solution of the upper component, which satisfies the Schrödinger equation (we assume that the upper component has a normalizable zero energy state) $H_-\psi = E\psi$, where

$$H_- = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) \tag{6}$$

with

$$V(x) = W(x)^2 - W'(x). \tag{7}$$

From the knowledge of shape invariant SUSY potentials, it was found that there are four allowed forms of shape invariant $\bar{W}(x)$ for which the spectrum of the Pauli equation can be algebraically written down [2]. One of the four forms gives rise to a uniform magnetic field.

The SUSY structure of the Pauli equation can be made use of in a different way, namely, in its close connection with quasi-exact solvability [11, 12]. We shall give a brief description of this connection below.

Consider a system described by equation (6). We shall look for $V(x)$ such that the system is QES. According to the theory of QES models, one first makes an imaginary gauge transformation on the wavefunction $\psi(x)$

$$\psi(x) = \phi(x)\,e^{-g(x)} \tag{8}$$

where $g(x)$ is the gauge function. The function $\phi(x)$ satisfies

$$-\frac{d^2\phi(x)}{dx^2} + 2g'\frac{d\phi(x)}{dx} + [V(x) + g'' - g'^2]\phi(x) = E\phi(x). \tag{9}$$

For physical systems which we are interested in, the phase factor $\exp(-g(x))$ is responsible for the asymptotic behaviour of the wavefunction so as to ensure normalizability. The function $\phi(x)$ satisfies a Schrödinger equation with a gauge transformed Hamiltonian

$$H_G = -\frac{d^2}{dx^2} + 2W_0(x)\frac{d}{dx} + \left[V(x) + W_0' - W_0^2\right] \tag{10}$$

where $W_0(x) = g'(x)$. Now if $V(x)$ is such that the quantal system is QES, that means the gauge-transformed Hamiltonian $H_G$ can be written as a quadratic combination of the generators $J^a$ of some Lie algebra with a finite-dimensional representation. Within this finite-dimensional Hilbert space the Hamiltonian $H_G$ can be diagonalized, and, therefore, a finite number of eigenstates is solvable. For one-dimensional QES systems the most general Lie algebra is $sl(2)$ [4–11]. Hence, if equation (10) is QES then it can be expressed as

$$H_G = \sum C_{ab}J^a J^b + \sum C_a J^a + \text{constant} \tag{11}$$

where $C_{ab}$, $C_a$ are constant coefficients, and the $J^a$ are the generators of the Lie algebra $sl(2)$ given by

$$\begin{aligned} J^+ &= z^2\frac{d}{dz} - Nz \\ J^0 &= z\frac{d}{dz} - \frac{N}{2} \qquad N = 0, 1, 2, \ldots \\ J^- &= \frac{d}{dz}. \end{aligned} \tag{12}$$

Here the variables $x$ and $z$ are related by $z = h(x)$, where $h(\cdot)$ is some (explicit or implicit) function. The value $j = N/2$ is called the weight of the differential representation of $sl(2)$ algebra, and $N$ is the degree of the eigenfunctions, which are polynomials in a $(N+1)$-dimensional Hilbert space.

The requirement in equation (11) fixes $V(x)$ and $W_0(x)$, and $H_G$ will have an algebraic sector with $N+1$ eigenvalues and eigenfunctions. In this sector the eigenfunction has the general form

$$\psi = (z - z_1)(z - z_2)\cdots(z - z_N)\,\exp\left(-\int^z W_0(x)\,dx\right) \tag{13}$$

where $z_i$ $(i = 1, 2, \ldots, N)$ are $N$ parameters that can be determined from the eigenvalue equations, namely, the Bethe ansatz equations corresponding to the QES problem [9, 13]. One can rewrite equation (13) as

$$\psi = \exp\left(-\int^z W_N(x, \{z_i\})\,dx\right) \tag{14}$$

and

$$W_N(x, \{z_i\}) = W_0(x) - \sum_{i=1}^{N} \frac{h'(x)}{h(x) - z_i}. \tag{15}$$

There are $N + 1$ possible functions $W_N(x, \{z_i\})$ for the $N + 1$ sets of eigenfunctions $\psi$. It is easy to check that $W_N$ satisfies the Ricatti equation [11, 12]

$$W^2 - W' = V - E \qquad (16)$$

where $E$ is the eigenenergy corresponding to the eigenfunction $\psi$ given in equation (13) for a particular set of parameters $\{z_i\}$. Equation (16) shows the connection between SUSY and QES problems.

From equations (6), (7) and (16), it is clear how one should proceed to determine the magnetic fields so that the Pauli equation becomes QES based on $sl(2)$: one need only determine the superpotentials $W(x)$ according to equation (16) from the QES potentials $V(x)$ classified by Turbiner [5]. This is easily done by observing that the superpotential $W_0$ corresponding to $N = 0$ is related to the gauge function $g(x)$ associated with a particular class of QES potential $V(x)$ by $g'(x) = W_0(x)$. Once $W_0$ is obtained, then $B_0 = -W_0'(x)$ is the required magnetic field that allows the weight zero ($j = N = 0$) state to be known in that class. But this state is just the ground state, and hence we have not gone beyond the result of [1]. What is more interesting is to obtain higher weight states (i.e. $j > 0$), which will include excited states. For weight $j$ ($N = 2j$) states, this is achieved by forming the superpotential $W_N(x, \{z_i\})$ according to equation (15). Of the $N + 1$ possible sets of solutions of the Bethe ansatz equations, the set of roots $\{z_1, z_2, \ldots, z_N\}$ to be used in equation (15) is chosen to be the set for which the energy of the corresponding state is the lowest (usually it is the ground state). The required magnetic field which gives rise to the $N + 1$ solvable states is then obtained as $B_N = -W_N'$. From the table in [5] it is easily seen that only six classes need be considered, namely class I to class VI. Classes VII to IX are excluded because these are QES systems with basic variables defined only on the half-line $(0, \infty)$, while class X corresponds to periodic potentials giving rise to nonnormalizable wavefunctions. Below we shall illustrate our construction of QES magnetic fields through the class I and II QES systems, which serve as representative examples of two different types of QES problems.

### 2.1. Class I

According to Turbiner's classification, the QES potential belonging to class I has the form[3]

$$V_N(x) = a^2 \, e^{-2\alpha x} - a[\alpha(2N + 1) + 2b_k] \, e^{-\alpha x} + c(2b_k - \alpha) \, e^{\alpha x} + c^2 \, e^{2\alpha x} + b_k^2 - 2ac. \qquad (17)$$

Here $b_k \equiv b + k$ with constant $b$. Without loss of generality, we assume $\alpha, a, c > 0$ for definiteness. The corresponding gauge function $g(x)$ is given by

$$g(x) = \frac{a}{\alpha} \, e^{-\alpha x} + \frac{c}{\alpha} \, e^{\alpha x} + b_k x. \qquad (18)$$

One should always keep in mind that the parameters selected must ensure convergence of the function $\exp(-g(x))$ in order to guarantee normalizability of the wavefunction (this is generally not required by the mathematicians). We have also added the constant $\left(b_k^2 - 2ac\right)$ in $V_N$ so that for $j = 0$, the energy of the ground state is zero ($E = 0$). This is not necessary, but it allows the results for $j = 0$ and $j > 0$ to be presented in a unified way. The potential $V(x)$ that gives the ground state is generated by

$$\begin{aligned} V(x) &= V_0 - E \\ &= W_0^2 - W_0' \end{aligned} \qquad (19)$$

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[3] We follow the notation used in [5] as closely as possible for the three classes discussed in this paper. We also take this opportunity to correct some typographical errors in that paper.

with

$$W_0(x) = g'(x)$$
$$= -a\,e^{-\alpha x} + c\,e^{\alpha x} + b + k. \tag{20}$$

The corresponding magnetic field is given by

$$B_0 = -W'_0(x)$$
$$= -a\alpha\,e^{-\alpha x} - c\alpha\,e^{\alpha x}. \tag{21}$$

To obtain magnetic fields and the corresponding potentials which admit solvable states with higher weights $j$, we must first derive the Bethe ansatz equations. To this end, let us perform the change of variable $z = h(x) = \exp(-\alpha x)$. Equation (9) then becomes

$$-\alpha z^2 \frac{d^2\phi(z)}{dz^2} + [2az^2 - (2b_k + \alpha)z - 2c]\frac{d\phi(z)}{dz} + \left[-2aNz - \frac{E}{\alpha}\right]\phi(z) = 0 \tag{22}$$

which can be written as a quadratic combination of the $sl(2)$ generators $J^+$, $J^-$ and $J^0$ as

$$T_{\mathrm{I}}\phi = 0$$
$$T_{\mathrm{I}} = -\alpha J^+ J^- + 2a J^+ - [\alpha(N+1) + 2b_k]J^0 - 2c J^- + \text{constant.} \tag{23}$$

For $N > 0$, there are $N + 1$ solutions which include excited states. Assuming $\phi(z) = \prod_{i=1}^{N}(z - z_i)$ in equation (22), one obtains the Bethe ansatz equations which determine the roots $z_i$

$$2az_i^2 - (2b_k + \alpha)z_i - 2c - 2\alpha \sum_{l \neq i} \frac{z_i^2}{z_i - z_l} = 0 \qquad i = 1, \dots, N \tag{24}$$

and the equation which gives the energy in terms of the roots $z_i$

$$E = 2\alpha c \sum_{i=1}^{N} \frac{1}{z_i}. \tag{25}$$

Each set of $\{z_i\}$ determines a QES energy $E$ with the corresponding polynomial $\phi$.

As an example, consider the $j = 1/2$ case with $N = 1$ and $\phi(z) = z - z_1$. There are two solutions. From equation (24), one sees that the root $z_1$ satisfies

$$2az_1^2 - (2b_k + \alpha)z_1 - 2c = 0 \tag{26}$$

which gives two solutions

$$z_1^{\pm} = \frac{(2b_k + \alpha) \pm \sqrt{(2b_k + \alpha)^2 + 16ac}}{4a}. \tag{27}$$

The corresponding energy is

$$E^{\pm} = 2\alpha c \frac{1}{z_1}$$
$$= -\frac{\alpha}{2}\left[(2b_k + \alpha) \mp \sqrt{(2b_k + \alpha)^2 + 16ac}\right]. \tag{28}$$

For the parameters assumed here, the solution with root $z_1^- = -|z_1^-| < 0$ gives the ground state, while that with root $z_1^+ > 0$ gives the first excited state. The superpotential $W_1$ is constructed according to equation (15)

$$W_1(x) = W_0 - \frac{h'(x)}{h(x) - z_1^-}$$
$$= -a\,e^{-\alpha x} + c\,e^{\alpha x} + \frac{\alpha}{1 + |z_1^-|\,e^{\alpha x}} + b + k. \tag{29}$$

This gives the magnetic field

$$
\begin{aligned}
B_1 &= -W_1'(x) \\
&= -a\alpha\,\mathrm{e}^{-\alpha x} - c\alpha\,\mathrm{e}^{\alpha x} + \alpha^2\,\frac{|z_1^-|\,\mathrm{e}^{\alpha x}}{(1+|z_1^-|\,\mathrm{e}^{\alpha x})^2}
\end{aligned}
\tag{30}
$$

and the potential that allows these two solvable states is

$$
\begin{aligned}
V(x) &= W_1^2 - W_1' \\
&= V_1 - E^-.
\end{aligned}
\tag{31}
$$

With this potential, the ground state and the excited state have energy $E = 0$ and $E = E^+ - E^- = \alpha\sqrt{(2b_k + \alpha)^2 + 16ac}$, respectively. Our construction, based on the connection between SUSY and QES systems, always sets the energy of the lowest energy state to zero.

This example should convey the general ideas of our construction. QES potentials and magnetic fields for higher degree $N$ are obtained in the same manner. We note that even for higher values of $N$ the equation (24) still remains an algebraic equation whose solutions can always be found albeit maybe numerically. But even then the system remains a QES one.

We mention here that QES systems belonging to classes IV and VI can be considered in a similar manner.

## 2.2. Class II

We shall consider one more class of QES potential, namely, class II of Turbiner's classification. The general procedure is same as that applied to class I. But unlike classes I, IV and VI, which are called the first type QES problems, classes II, III and V belong to the second type. In the first type QES problems, $N + 1$ eigenstates are solvable for a fixed potential with a fixed degree $N$. For the second type, on the other hand, there are $N + 1$ QES potentials differing by the values of parameters and having the same eigenvalue of the $i$th eigenstate in the $i$th potential. For our present problem, each potential corresponds to a magnetic field. Below we shall demonstrate this using class II potentials.

The general form of class II QES potential is

$$
\begin{aligned}
V_N(x, \lambda) = d^2\,\mathrm{e}^{-4\alpha x} + 2ad\,\mathrm{e}^{-3\alpha x} + [a^2 + 2d(b_k - \alpha(N+1))]\,\mathrm{e}^{-2\alpha x} \\
+ (2ab_k - \alpha a + \lambda)\,\mathrm{e}^{-\alpha x} + b_k^2.
\end{aligned}
\tag{32}
$$

The gauge function is

$$
g(x) = \frac{d}{2\alpha}\,\mathrm{e}^{-2\alpha x} + \frac{a}{\alpha}\,\mathrm{e}^{-\alpha x} - b_k x.
\tag{33}
$$

As mentioned before, the parameters must be so chosen as to guarantee the normalizability of the wavefunction. For definiteness, we assume $\alpha, d > 0$ and $b_k = b + k < 0$.

Letting $z = h(x) = \exp(-\alpha x)$, the equation of $\phi(z)$ is

$$
-\alpha z\,\frac{\mathrm{d}^2\phi(z)}{\mathrm{d}z^2} + [2dz^2 + 2az + 2b_k - \alpha]\frac{\mathrm{d}\phi(z)}{\mathrm{d}z} + \left[-2dNz + \frac{\lambda}{\alpha} - \frac{E}{z\alpha}\right]\phi(z) = 0.
\tag{34}
$$

The differential operator in equation (34) can also be written as

$$
T_{\mathrm{II}} = -\alpha J^0 J^- + 2d J^+ + 2a J^0 - \left[\alpha\left(\frac{N}{2}+1\right) - 2b_k\right] J^- + \text{constant}.
\tag{35}
$$

The energy $E$ and the parameter $\lambda$ are given by

$$
\begin{aligned}
E &= 0 \\
\lambda &= \alpha(2b_k - \alpha)\sum_{i=1}^{N}\frac{1}{z_i}
\end{aligned}
\tag{36}
$$

where the $z_i$ are to be solved from the Bethe ansatz equations

$$dz_i^2 + az_i + b_k - \frac{\alpha}{2} - \alpha \sum_{l \neq i} \frac{z_i}{z_i - z_l} = 0 \qquad i = 1, \ldots, N. \tag{37}$$

The required magnetic field is again given by the roots $z_k$ through equation (15). For $N = 0$, one has $\lambda = 0$.

So far everything appears to be the same as for class I. The main point is to note that $V_N(x, \lambda)$ is a function of the parameter $\lambda$ as well as $N$, and $\lambda$ is determined from the roots $\{z_1, z_2, \ldots, z_N\}$. For a fixed $N$ there are $N + 1$ possible sets of the roots. Therefore, one can construct $N + 1$ possible potentials $V_N^{(m)}(x)$ ($m = 0, 1, \ldots, N$) for equation (6) according to

$$\begin{aligned} V_N^{(m)} &= \left( W_N^{(m)} \right)^2 - \left( W_N^{(m)} \right)' \\ &= V_N(x, \lambda^{(m)}) - E \qquad E = 0. \end{aligned} \tag{38}$$

Here $\lambda^{(m)}$ is the parameter evaluated using the $m$th set of roots of the Bethe ansatz equations in equation (36), and $W_N^{(m)}$ is obtained from equation (15) using the same set of roots. We recall here that in class I discussed previously, the superpotential $W_N$ was calculated using the set of roots which gives the lowest energy, but here all the $N + 1$ sets of roots have to be used. For each potential $V_N^{(m)}$ only one state is solved (by the $m$th set of roots) with the same energy $E = 0$. And each potential $V^{(m)}$ corresponds to a magnetic field $B_N^{(m)} = - \left( W_N^{(m)} \right)'$. For the family of potentials, only some (generally one) potentials have the ground state solved, while for others the solvable state is an excited state. In other words, we have a family of magnetic fields for which the Pauli equation is solvable for one level with the same energy.

To illustrate these points, let us now give two examples. Consider first the case for $N = 0$, giving only the ground state $\psi(x) = \text{const.} \times \exp(-g(x)$ with $g(x)$ being given by equation (33). The energy of this state is $E = 0$. In this case the parameter $\lambda$ is $\lambda = 0$, and the QES potential that gives rise to this solvable ground state is

$$V(x) = V_0^{(0)} = V_0(x, \lambda = 0) \tag{39}$$

which according to equation (38) is generated by the superpotential

$$\begin{aligned} W_0^{(0)}(x) &= g'(x) \\ &= -d\, e^{-2\alpha x} - a\, e^{-\alpha x} - b - k. \end{aligned}$$

The corresponding magnetic field is

$$\begin{aligned} B_0 &= -W_0^{(0)\prime}(x) \\ &= -2d\alpha\, e^{-2\alpha x} - a\alpha\, e^{-\alpha x}. \end{aligned} \tag{40}$$

Now we come to the case for $N = 1$. The QES wavefunction is $\psi = (z - z_1) \exp(-g)$, and the energy of the state is $E = 0$. The root $z_1$ is solved from the Bethe ansatz equation (37)

$$dz_1^2 + az_1 + b_k - \frac{\alpha}{2} = 0. \tag{41}$$

We recall here that we have assumed $\alpha, d > 0$ and $b_k = b + k < 0$. Equation (41) gives two solutions

$$z_1^{(0,1)} = \frac{-a \mp \sqrt{a^2 + 4d(|b_k| + \alpha/2)}}{2d} \tag{42}$$

where $z_1^{(0)}$ $\left( z_1^{(1)} \right)$ corresponds to the solution given by the minus (plus) sign. Since $z_1^{(0)} < 0$, the state $\psi$ constructed with $z_1^{(0)}$ is the ground state, while that with $z_1^{(1)}$ is the first excited state. According to equation (38), the ground state is the only QES state for the system with

potential $V_1^{(0)} = V_1(x, \lambda^{(0)})$, and the excited state is the only QES state for the potential $V_1^{(1)} = V_1(x, \lambda^{(1)})$, where the parameters $\lambda^{(0,1)}$ are given by

$$\lambda^{(0,1)} = -\frac{\alpha\,(2|b_k| + \alpha)}{z_1^{(0,1)}}. \tag{43}$$

These two potentials are generated by the superpotentials

$$\begin{aligned} W_1^{(m)}(x) &= W_0^{(0)} - \frac{h'(x)}{h(x) - z_1^{(m)}} \\ &= -d\,e^{-2\alpha x} - a\,e^{-\alpha x} + \frac{\alpha}{1 - z_1^{(m)}\,e^{\alpha x}} - b - k \qquad m = 0, 1 \end{aligned} \tag{44}$$

with the corresponding magnetic fields being

$$\begin{aligned} B_1^{(m)} &= -W_1^{(m)\prime}(x) \\ &= -2d\alpha\,e^{-2\alpha x} - a\alpha\,e^{-\alpha x} - \alpha^2\frac{z_1^{(m)}\,e^{\alpha x}}{\left(1 - z_1^{(m)}\,e^{\alpha x}\right)^2} \qquad m = 0, 1. \end{aligned} \tag{45}$$

The point is to note that the energy of the ground state for the potential $V_1^{(0)}$ and that of the first excited state for $V_1^{(1)}$ are both equal to zero, i.e. $E = 0$.

The cases for classes III and V are the same as the present one. We shall not repeat the arguments here.

## 3. Magnetic field in symmetric gauge

We now consider the same problem in the symmetric gauge

$$A_x = yf(r) \qquad A_y = -xf(r) \tag{46}$$

where $r^2 = x^2 + y^2$. The magnetic field $B_z = B$ is then given by

$$B(x, y) = -2f(r) - rf'(r). \tag{47}$$

The Pauli Hamiltonian is

$$H = -\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) + r^2 f^2 - 2fL_z - (2f + rf')\sigma_z. \tag{48}$$

Here $L_z$ is the $z$-component of the orbital angular momentum, and $f' = df/dr$. We assume the wavefunctions to have the form

$$\Psi(t, \mathbf{x}) = \frac{1}{\sqrt{r}}\psi_m(r, \varphi) \tag{49}$$

with

$$\psi_m(r, \varphi) = \begin{pmatrix} R_1(r)\,e^{im\varphi} \\ R_2(r)\,e^{i(m+1)\varphi} \end{pmatrix} \tag{50}$$

with integral number $m$. The function $\psi_m(r, \varphi)$ is an eigenfunction of the conserved total angular momentum $J_z = L_z + \sigma_z/2$ with eigenvalue $J = m + 1/2$. The components $R_1$ and $R_2$ satisfy

$$\left[-\frac{d^2}{dr^2} + r^2 f^2 - 2f(m + 1) - rf' + \frac{m^2 - \frac{1}{4}}{r^2}\right] R_1(r) = E R_1(r) \tag{51}$$

and

$$\left[ -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + r^2 f^2 - 2fm + rf' + \frac{(m+1)^2 - \frac{1}{4}}{r^2} \right] R_2(r) = E R_2(r) \tag{52}$$

where $E$ is the energy.

The gauge function $g(r)$ for equation (51), which accounts for the asymptotic behaviour of the system, has the general form

$$g(r) = \int^r \rho f(\rho) \, \mathrm{d}\rho - \gamma \ln r \qquad \gamma = |m| + 1/2. \tag{53}$$

The corresponding superpotential $W(r)$ is

$$W(r) = g'(r)$$
$$= rf(r) - \frac{\gamma}{r}. \tag{54}$$

One can check that the potentials in equations (51) and (52) are given by $V_-$ and $V_+$, respectively, where $V_\mp = W^2 \mp W'$ for positive $m \geqslant 0$. This again shows the SUSY structure of the Pauli equation. Hence, the procedure presented in the last section can also be applied in this case (for $m \geqslant 0$).

Our task is to find the form of $f(r)$ such that the Pauli equation is QES. It is seen that both equations (51) and (52) are in the Schrödinger form. So one could try to find $f(r)$ that would fit equations (51) and (52) into the forms classified in [5]. As before, we shall only consider the upper component $R_1$. The lower component can be obtained by SUSY. We found that there exist three forms of magnetic fields which make the Pauli equation QES. These three forms of magnetic fields give QES potentials that belong to classes VII, VIII and IX of Turbiner's classification. Below we shall discuss only the case for class VII. The other two classes can be considered in a similar manner.

### 3.1. Class VII

By inspection one finds that if we let $f(r) = f_0(r) = ar^2 + b$ ($a > 0$, $b$ are constants), then equation (51) belongs to class VII of Turbiner's classification with $N = 0$. With this form of the function $f$, the potential in equation (51) is

$$V(r) = a^2 r^6 + 2abr^4 + [b^2 - 2a(m+2)]r^2 + \gamma(\gamma - 1)r^{-2} - 2b(m+1). \tag{55}$$

The magnetic field is $B_0 = -4ar^2 - 2b$. The general potential in class VII has the form

$$V_N(r) = a^2 r^6 + 2abr^4 + [b^2 - a(4N + 2\gamma + 3)]r^2 + \gamma(\gamma - 1)r^{-2} - b(2\gamma + 1). \tag{56}$$

Comparing equations (55) and (56) one concludes that the potential (55) does belong to class VII with $N = 0$ and for $m \geqslant 0$.

As in the asymmetric case, we assume $R_1 = \exp(-g(r))\phi$, then $\phi$ satisfies the same equation (9) with all the derivatives now being with respect to the variable $r$ instead of $x$. With the choice

$$g(r) = \frac{a}{4}r^4 + \frac{b}{2}r^2 - \gamma \ln r \tag{57}$$

one can check that $V(r)$ in equation (55) is generated by $W_0(r) \equiv g'(r)$ in the form $V = W_0^2 - W_0'$. Hence, the method used in the asymmetric gauge can also be applied here to generate magnetic fields which allow for QES potentials with higher weight. To proceed, we need to obtain the Bethe ansatz equations for $\phi$.

Letting $z = h(r) = r^2$, equation (9) becomes

$$-4z\frac{d^2\phi(z)}{dz^2} + [4az^2 + 4bz - 2(2\gamma + 1)]\frac{d\phi(z)}{dz} - [4aNz + E]\phi(z) = 0. \tag{58}$$

In terms of the $sl(2)$ generators $J^+$, $J^-$ and $J^0$, the differential operator in equation (58) can be written as

$$T_{\text{VII}} = -4J^0 J^- + 4aJ^+ + 4bJ^0 - 2(N + 2\gamma + 1)J^- + \text{constant}. \tag{59}$$

For $N = 0$, the energy is $E = 0$. For higher $N > 0$ and $\phi(r) = \prod_{i=1}^{N}(z - z_i)$, the function $f(r) = f_N(r)$ is obtained from equations (15) and (54)

$$f_N(r) = f_0(r) - \frac{1}{r}\sum_{i=1}^{N}\frac{h'(r)}{h(r) - z_i}. \tag{60}$$

For the present case, the roots $z_i$ are found from the Bethe ansatz equations

$$2az_i^2 + 2bz_i - (2\gamma + 1) - \sum_{l \neq i}\frac{z_i}{z_i - z_l} = 0 \qquad i = 1, \ldots, N \tag{61}$$

and the energy in terms of the roots $z_i$ is

$$E = 2(2\gamma + 1)\sum_{i=1}^{N}\frac{1}{z_i}. \tag{62}$$

For $N = 1$ the roots $z_1$ are

$$z_1^{\pm} = \frac{-b \pm \sqrt{b^2 + 2a(2\gamma + 1)}}{2a}. \tag{63}$$

The energies are

$$E^{\pm} = 2(b \pm \sqrt{b^2 + 2a(2\gamma + 1)}). \tag{64}$$

For $a > 0$, the root $z_1^- = -|z_1^-| < 0$ gives the ground state. With this root, one gets the superpotential

$$W_1(r) = ar^3 + br - \frac{2r}{r^2 + |z_1^-|} - \frac{\gamma}{r}. \tag{65}$$

The QES potential appropriate for the Pauli problem is

$$V(x) = W_1^2 - W_1'$$
$$= V_1 - E^-. \tag{66}$$

With this choice of the potential, the ground state and the excited state have energy $E = 0$ and $E = E^+ - E^- = 4\sqrt{b^2 + 2a(2\gamma + 1)}$, respectively. The magnetic field $B_1$ is calculated from equation (47) using the function

$$f_1(r) = \frac{1}{r}\left[W_1(r) + \frac{\gamma}{r}\right]$$
$$= ar^2 + b - \frac{2}{r^2 + |z_1^-|} \tag{67}$$

which gives

$$B_1(r) = -4ar^2 - 2b + \frac{4|z_1^-|}{(r^2 + |z_1^-|)^2}. \tag{68}$$

Just as class I, class VII is also of the first type. On the other hand, classes VIII and IX belong to the second type. We will not repeat the discussions here.

## 4. Summary and discussions

In this paper an attempt to give a QES generalization of the result of Aharonov and Casher is presented. We have given a general procedure for determining possible (nonuniform) magnetic fields such that the Pauli equation becomes QES based on the $sl(2)$ algebra. This procedure makes full use of the close connection between QES systems and SUSY. Of the ten classes of $sl(2)$-based one-dimensional QES systems, we have found that nine classes allow such a construction. It would be interesting to extend our procedure to the Dirac equation.

The Pauli equation is supersymmetric owing to the fact that the gyromagnetic ratio is two, i.e. $g = 2$. We would like to mention that recently it was realized [15, 16] that if one changes $g$ to some unphysical values $g = 2n$ ($n$ positive integers), then for magnetic fields of special exponential and quadratic forms, the generalized Pauli equation could possess a new type of supersymmetry [17–19]. This is the nonlinear generalization of the usual supersymmetry, and is given the name 'nonlinear holomorphic supersymmetry' in [15, 16, 19] or '$n$-fold supersymmetry' in [18]. It is characterized by a nonlinear superalgebra among the supercharges and the Hamiltonian, and the anticommutator of the supercharges is a polynomial of the Hamiltonian. The usual SUSY can be viewed as a special case, namely, the $n = 1$ case of the $n$-fold SUSY. Soon after its discovery, the $n$-fold SUSY was shown to be closely related to quasi-exact solvability [18, 19]. For the generalized Pauli equations considered in [15, 16], the weight $j = N/2$ characterizing the quasi-exact solvability of the system is given in terms of the number $n$ of the $n$-fold SUSY and some parameters of the system. The authors of [16] found certain duality transformations which mix the number $n$ and the parameters to give different values of $N$. These duality transformations thus connect different sectors of the generalized Pauli equations. From a mathematical point of view, quasi-exact solvability of the generalized Pauli equation is an interesting subject to be further explored. It is worth mentioning that the main difference between the generalized Pauli equation on a plane considered in [15] and the system considered by us is that in [15] the weight $j = N/2$ is related to the $n$-fold SUSY by $n = 2j + 1 = N + 1$, while in our case $n$ is always one (i.e. $n = 1$) and $N$ can be chosen arbitrarily ($N = 0, 1, 2, \ldots$). Hence when the system in [15] is reduced to our case (by setting $n = 1$), the only QES state that is retained is the ground state (corresponding to $N = 0$). Furthermore, since the number $N$ in our case is an arbitrarily chosen number, the kind of duality transformation obtained in [16] does not exist in our system.

Finally we mention a few things about the degeneracy of the energy levels. First we note that the Hamiltonians $H_\pm$ are SUSY partners (since they are built from *nodeless* superpotentials) and thus $H_+$ shares all the levels of $H_-$ except the zero energy state. Therefore, all the levels of $H_-$ are doubly degenerate except the zero energy level. This is in agreement with the results of [1]. We now come to the question of degeneracy of the levels within one component, namely, $H_-$. Since in all the cases considered in this paper the magnetic fields are nonuniform, so according to [20] the excited states are nondegenerate. This behaviour of the excited states is in contrast to the ground state which is always degenerate with the degeneracy depending on the magnetic flux.

## References

[1]  Aharonov Y and Casher A 1979 *Phys. Rev.* A **19** 2461
[2]  Stanciu G 1967 *J. Math. Phys.* **8** 2043
     Cooper F, Khare A, Musto R and Wipf A 1988 *Ann. Phys., NY* **187** 1
     Cooper E, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267 (section 10)
[3]  Andrianov A A and Ioffe M V 1988 *Phys. Lett.* B **205** 507
[4]  Turbiner A and Ushveridze A G 1987 *Phys. Lett.* A **126** 181
[5]  Turbiner A V 1988 *Commun. Math. Phys.* **118** 467
[6]  González A, Kamran N and Olver P J 1993 *Commun. Math. Phys.* **153** 117
     González A, Kamran N and Olver P J 1994 *Contemp. Math. Phys.* **160** 113
[7]  Turbiner A V 1994 *Contemp. Math.* **160** 263
     Post G and Turbiner A V 1995 *Russ. J. Math. Phys.* **3** 113
[8]  Shifman M A and Turbiner A V 1989 *Commun. Math. Phys.* **126** 347
     Shifman M A 1994 *Contemp. Math.* **160** 237
[9]  Ushveridze A G 1988 *Sov. Phys.-Lebedev Inst. Rep.* **2** 50 54
     Ushveridze A G 1994 *Quasi-Exactly Solvable Models in Quantum Mechanics* (Bristol: Institute of Physics Publishing)
[10]  Kamran N and Olver P 1990 *J. Math. Anal. Appl.* **145** 342
[11]  Shifman M A 1989 *Int. J. Mod. Phys.* A **4** 2897
      Shifman M A 1989 *Int. J. Mod. Phys.* A **4** 3305
[12]  Roy P, Roy B and Roychoudhury R 1989 *Phys. Lett.* **139** 427
[13]  Ho C-L and Khalilov V R 2000 *Phys. Rev.* A **61** 032104
      Chiang C-M and Ho C-L 2001 *Phys. Rev.* A **63** 062105
      Chiang C-M and Ho C-L 2002 *J. Math. Phys.* **43** 43
[14]  Tkachuk V M and Vakarchuk S I 2002 *J. Phys. Stud.* **6** 147
[15]  Klishevich S M and Plyushchay M S 2001 *Nucl. Phys.* B **616** 403
[16]  Klishevich S M and Plyuschay M 2002 *Nucl. Phys.* B **640** 481
[17]  Andrianov A A, Ioffe M V and Spiridonov V P 1993 *Phys. Lett.* A **174** 273
      Bagrov V G and Samsonov B F 1995 *Theor. Math. Phys.* **104** 1051
      Klishevich S M and Plyushchay M S 1999 *Mod. Phys. Lett.* A **14** 2739
      Plyushchay M S 2000 *Int. J. Mod. Phys.* A **15** 3679
[18]  Aoyama H, Kikuchi H, Okouchi I, Sato M and Wada S 1999 *Nucl. Phys.* B **553** 644
      Aoyama H, Sato M and Tanaka T 2001 *Phys. Lett.* B **503** 423
      Aoyama H, Sato M and Tanaka T 2001 *Nucl. Phys.* B **619** 105
[19]  Klishevich S M and Plyushchay M S 2001 *Nucl. Phys.* B **606** 583
[20]  Grosse H, Martin A and Stubbe J 1993 *Phys. Lett.* A **181** 7